三次系统极限环及其稳定和分岔的一种算法*

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摘 要： 引进适当的参数，求出该参数近似为零时系统的解答；以此解答为初值，给参数以小增量（即参数摄动）；将平面三次多项式微分系统极限环相图的 x 坐标假设为广义谐函数；将 y 坐标和频率作富氏展开；相应于参数的增量，得到极限环振幅、偏心距以及 y 坐标和频率的富氏系数的增量；用谐波平衡法得到以这些增量为独立变量的线性代数方程组；求解该方程组，得到各相关增量；以这些增量与初值的和为下一参数增量步骤相应的初值，重复上述过程，直至参数还原至原系统为止，从而得到极限环及其频率、周期、稳定性指标，以及极限环关于参数分岔曲线的近似解析表达式。文末给出算例。

关键词： 平面三次多项式微分系统；极限环；稳定；分岔；算法
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An Algorithm of Stability and Bifurcation of Limit Cycles for Cubic System

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Abstract: With a suitable parameter, the solution of the system is solved as this parameter equaled zero. This solution is taken as the initial value, and the parameter is given a small increment. The x coordinate of limit cycle phase portraits for planar cubic polynomial differential systems are supposed as the generalized harmonic function. And the y coordinate and the frequency of limit cycle are expanded as Fourier series. Corresponding to the increments of the parameter, the increment of the amplitude, eccentricity and the Fourier coefficients of y coordinate and the frequency of limit cycle are obtained. The linear algebra equations about these increments are got with harmonic balance. Solving these equations, these increments are obtained. The procedure is repeated with the initial value of the next step as the sum of the increments and the initial value, until the parameter is returned to original state. And then the approximate analytical expressions of frequency, periodic, stability index and bifurcation of limit cycles about the parameter are calculated. An example is shown at the end.

Key words: planar cubic polynomial differential systems; limit cycle; stability; bifurcation; algorithm

许多动力学与控制问题可以归结为如下平面三次多项式微分系统极限环的研究：

$$\begin{aligned}\dot{x} &= \alpha_1 x + \alpha_2 y + \alpha_3 x^2 + \alpha_4 xy + \alpha_5 y^2 + \\ &\quad \alpha_6 x^3 + \alpha_7 x^2 y + \alpha_8 xy^2 + \alpha_9 y^3 \\ \dot{y} &= \beta_1 x + \beta_2 y + \beta_3 x^2 + \beta_4 xy + \beta_5 y^2 + \\ &\quad \beta_6 x^3 + \beta_7 x^2 y + \beta_8 xy^2 + \beta_9 y^3 \end{aligned} \quad (1)$$

其中，$\alpha_i, \beta_i (i = 1,2,\cdots,9)$ 是实常量，$(\dot{\ }) = \frac{d(\ )}{dt}$（下同）。研究表明，系统（1）极限环具有奇妙的多环性，这给工程技术界予困惑：系统到底最多有多少个极限环？这就是著名的 D. Hilbert 第16问题

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的第二部分，历经学者一百余年研究[1-5]，目前尚无答案。本文暂不涉及这一难题。多项式系统极限环的定量表示已有相关研究结果[6-13]。本文利用摄动－增量法的思想[6]，给出形如系统（1）的一般三次多项式微分系统极限环算法近似解析表达式，具体做法是：引进适当的参数，求该参数近似为零时系统的解答；以此解答为初值，给参数予小增量（即参数摄动），将 x 坐标假设为广义谐函数，将 y 坐标和频率作富氏展开；相应于参数的增量，得到极限环振幅、偏心距以及 y 坐标和频率的富氏系数的增量；用谐波平衡法得到以这些增量为独立变量的线性代数方程组；求解该方程组，得到各相关增量；以这些增量与初值的和作为下一参数增量步骤相应的初值，重复上述过程，直至参数的变化还原原系统为止，从而得到极限环及其频率、周期、稳定性指标，以及极限环关于参数分岔曲线的高精度的近似解析表达式。也许，诸如上述精度较高的定量解答，对揭示相关系统的定性性质，比如极限环的最少数目即著名的 Hilbert 第 16 问题第二部分，有所启示。文末给出算例，并与数值积分法的比较显示，本算法的精度良好。

1 极限环的算法

引进参数 $\varepsilon > 0$，将系统（1）改写为

$$\dot{x} = \alpha_2 y + \varepsilon X(x,y)$$
$$\dot{y} = g(x) + \varepsilon Y(x,y) \tag{2}$$

式（2）中

$$X(x,y) = \alpha_1 x + \alpha_3 x^2 + \alpha_4 xy + \alpha_5 y^2 + \alpha_6 x^3 + \alpha_7 x^2 y + \alpha_8 xy^2 + \alpha_9 y^3$$
$$Y(x,y) = \beta_2 y + \beta_4 xy + \beta_5 y^2 + \beta_7 x^2 y + \beta_8 xy^2 + \beta_9 y^3$$
$$g(x) = \beta_1 x + \beta_3 x^2 + \beta_6 x^3$$

引进变量替换

$$\dot{\varphi} = \omega(\varphi) \tag{3}$$

满足 $\omega(\varphi + 2\pi) = \omega(\varphi) > 0$。易知变换（3）不改变（2）的拓扑结构。于是式（2）变为

$$\omega x' = \alpha_2 y + \varepsilon X(x,y)$$
$$\omega y' = g(x) + \varepsilon Y(x,y) \tag{4}$$

式（4）中，$(\,)' = \dfrac{\mathrm{d}(\,)}{\mathrm{d}\varphi}$（下同）。设

$$x = a\cos\varphi + b \tag{5}$$

其中 a 是振幅，b 是偏心距。将式（5）代入式（4）第一式，可得

$$y = -(a\omega\sin\varphi + \varepsilon X)/\alpha_2 \tag{6}$$

将式（6）代入式（4）第二式，可得

$$\omega(a\omega\sin\varphi)' = -\varepsilon\omega X' - \alpha_2 g(x) - \alpha_2\varepsilon Y \tag{7}$$

式（7）两边同乘以 $a\sin\varphi$ 并对 φ 积分，得

$$\frac{1}{2}(a\omega\sin\varphi)^2 - \alpha_2[v(a\cos\varphi + b) - v(a+b)] - \varepsilon\int_0^{\varphi} f(a,b,y,\omega,\theta)\mathrm{d}\theta = 0 \tag{8}$$

这里，

$$v(x) = \frac{1}{2}\beta_1 x^2 + \frac{1}{3}\beta_3 x^3 + \frac{1}{4}\beta_6 x^4$$
$$f(a,b,y,\omega,\theta) = a(\omega X' + \alpha_2 Y)\sin\theta$$

式（8）中，分别取 $\varphi = \pi, 2\pi$，可得

$$\alpha_2[v(-a+b) - v(a+b)] - \varepsilon\int_0^{\pi} f\mathrm{d}\theta = 0 \tag{9}$$

$$\int_0^{2\pi} f\mathrm{d}\theta = 0 \tag{10}$$

改写式（6）为

$$\alpha_2 y + a\omega\sin\varphi + \varepsilon X = 0 \tag{11}$$

原则上，经变换（3），系统（1）极限环的相关量 a,b,y,ω 可由式（8）～（11）确定，但难以精确求解。下面以摄动增量及谐波平衡的思想近似求解之。

2 摄动增量及谐波平衡法

取 $\varepsilon \approx 0$，由式（5）及（8）～（11）得

$$x_0 = a_0\cos\varphi + b_0 \tag{12}$$

$$\omega_0 = \left\{\frac{2\alpha_2[v(a_0\cos\varphi + b_0) - v(a_0 + b_0)]}{a_0^2\sin^2\varphi}\right\}^{\frac{1}{2}} \tag{13}$$

$$v(-a_0 + b_0) - v(a_0 + b_0) = 0 \tag{14}$$

$$\int_0^{2\pi}[(\alpha_1 x_0 + \alpha_3 x_0^2 + \alpha_4 x_0 y_0 + \alpha_5 y_0^2 + \alpha_6 x_0^3 + \alpha_7 x_0^2 y_0 + \alpha_8 x_0 y_0^2 + \alpha_9 y_0^3)(\omega_0\sin\varphi)' - \alpha_2(\beta_2 y_0 + \beta_4 x_0 y_0 + \beta_5 y_0^2 + \beta_7 x_0^2 y_0 + \beta_8 x_0 y_0^2 + \beta_9 y_0^3)\sin\varphi]\mathrm{d}\varphi = 0 \tag{15}$$

$$y_0 = -a_0\omega_0\sin\varphi/\alpha_2 \tag{16}$$

联立式（12）～（16），可得 a_0, b_0。给参数 ε 予小增量 $\Delta\varepsilon_1$，相应地，a_0, b_0, y_0, ω_0 的增量分别为 $\Delta a, \Delta b, \Delta y, \Delta\omega$。令

$$\varepsilon_1 = \Delta\varepsilon_1, a_1 = a_0 + \Delta a, b_1 = b_0 + \Delta b,$$
$$y_1 = y_0 + \Delta y, \omega_1 = \omega_0 + \Delta\omega \tag{17}$$

分别展开 $y_0, \omega_0, \Delta y, \Delta\omega$ 为富氏展式：

$$y_0 = C_0 + \sum_{j=1}^{m}(C_j\cos j\varphi + D_j\sin j\varphi) \tag{18}$$

$$\omega_0 = P_0 + \sum_{j=1}^{m}(P_j\cos j\varphi + Q_j\sin j\varphi) \tag{19}$$

$$\Delta y = \Delta C_0 + \sum_{j=1}^{m}(\Delta C_j\cos j\varphi + \Delta D_j\sin j\varphi) \tag{20}$$

$$\Delta\omega = \Delta P_0 + \sum_{j=1}^{m}(\Delta P_j\cos j\varphi + \Delta Q_j\sin j\varphi) \quad (21)$$

将式（5）及（17）代入式（11）并略去关于Δa，$\Delta b,\Delta y,\Delta\omega$的二次以上的高次项，得

$$\Delta y(\alpha_2 + \varepsilon\frac{\partial X}{\partial y}) + \Delta\omega a\sin\varphi + \Delta a(\omega\sin\varphi + \varepsilon\frac{\partial X}{\partial a}) + \Delta b\varepsilon\frac{\partial X}{\partial b} = -\alpha_2 y - a\omega\sin\varphi - \varepsilon X \quad (22)$$

类似于式（22），由式（8）可得

$$-\varepsilon\int_0^{\varphi}\frac{\partial f}{\partial y}\Delta y\mathrm{d}\theta - \Delta\omega(a^2\omega\sin^2\varphi) - \varepsilon\int_0^{\varphi}\frac{\partial f}{\partial\omega}\Delta\omega\mathrm{d}\theta + \Delta a(\alpha_2\frac{\partial\tilde{v}}{\partial a} - a\omega^2\sin^2\varphi - \varepsilon\int_0^{\varphi}\frac{\partial f}{\partial a}\mathrm{d}\theta) + \Delta b(\alpha_2\frac{\partial\tilde{v}}{\partial b} - \varepsilon\int_0^{\varphi}\frac{\partial f}{\partial b}\mathrm{d}\theta) = -\frac{1}{2}(a\omega\sin\varphi)^2 - \alpha_2\tilde{v} + \varepsilon\int_0^{\varphi}f\,\mathrm{d}\theta \quad (23)$$

式（23）中，

$$\tilde{v}(a,b,\varphi) = v(a\cos\varphi + b) - v(a + b) \quad (24)$$

类似地，由式（10）得

$$-\varepsilon\int_0^{\pi}\frac{\partial f}{\partial b}\Delta y\mathrm{d}\theta - \varepsilon\int_0^{\pi}\frac{\partial f}{\partial\omega}\Delta\omega\mathrm{d}\theta + \Delta a[\alpha_2\frac{\partial\tilde{v}(a,b,\pi)}{\partial a} - \varepsilon\int_0^{\pi}\frac{\partial f}{\partial a}\mathrm{d}\theta] + \Delta b[\alpha_2\frac{\partial\tilde{v}(a,b,\pi)}{\partial b} - \varepsilon\int_0^{\pi}\frac{\partial f}{\partial b}\mathrm{d}\theta] = -\alpha_2\tilde{v}(a,b,\pi) + \varepsilon\int_0^{\pi}f\mathrm{d}\theta \quad (25)$$

式（25）中，

$$\tilde{v}(a,b,\pi) = -\frac{2}{3a}(a^2 + 3b^2 + 3a^2b + 3b^3)$$

由式（10），可得

$$\int_0^{2\pi}(\frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial\omega}\Delta\omega)\mathrm{d}\theta + \Delta a\int_0^{2\pi}\frac{\partial f}{\partial a}\mathrm{d}\theta + \Delta b\int_0^{2\pi}\frac{\partial f}{\partial b}\mathrm{d}\theta = -\int_0^{2\pi}f\mathrm{d}\theta \quad (26)$$

式（22）～（26）中的ε,a,b,y,ω即式(17)中的$\varepsilon_1,a_0,b_0,y_0,\omega_0$。将上述各式中所含的三角函数$\cos\varphi,\sin\varphi$的幂及乘积化为倍角的三角函数及其代数和形式，并利用谐波平衡原理，可得关于

$$\Delta C_0,\Delta C_1,\Delta C_2,\cdots,\Delta C_m;\Delta D_1,\Delta D_2,\cdots,\Delta D_m;$$
$$\Delta P_0,\Delta P_1,\Delta P_3,\cdots,\Delta P_m;\Delta Q_1,\Delta Q_2,\cdots,\Delta Q_m$$

的线性代数方程组

$$AZ = B \quad (27)$$

其中，

$$Z = [\Delta C_0,\cdots,\Delta C_m;\Delta D_1,\cdots,\Delta D_m;\Delta P_0,\cdots,\Delta P_m;\Delta Q_1,\cdots,\Delta Q_m;\Delta a,\Delta b]^T$$

$$B = [B_1 \quad B_2\cdots B_{4m+2} \quad B_{4m+3} \quad B_{4m+4}]^T$$

$$A = \begin{bmatrix} A_{11} & A_{12}\cdots & A_{1,4m+4} \\ A_{21} & A_{22}\cdots & A_{2,4m+4} \\ \cdots\cdots & & \\ A_{4m+2,1} & A_{4m+2,2}\cdots & A_{4m+2,4m+4} \\ A_{4m+3,1} & A_{4m+3,2}\cdots & A_{4m+3,4m+4} \\ A_{4m+4,1} & A_{4m+4,2}\cdots & A_{4m+4,4m+4} \end{bmatrix}$$

上式中，$A_{i,j},B_i(i,j = 1,2,\cdots,4m + 4)$的计算涉及一系列三角函数及三角级数乘法，限于篇幅，不一一罗列。

求解方程组（27）得到Z并将之代入式（20）、（21）及（17）。重复上述步骤，经参数ε第i次增量$\Delta\varepsilon_i$，相应得

$$\varepsilon_i = \varepsilon_{i-1} + \Delta\varepsilon_i, a_i = a_{i-1} + \Delta a, b_i = b_{i-1} + \Delta b,$$
$$y_i = y_{i-1} + \Delta y, \omega_i = \omega_{i-1} + \Delta\omega, i = 1,\cdots,k$$

直至$\varepsilon_k = 1$为止。此时的a_k,b_k,y_k,ω_k即为所求的a,b,y,ω。于是，绕奇点（0，0）的极限环、瞬时频率和周期的表达式分别为

$$\begin{cases} x = a\cos\varphi + b \\ y = C_0 + \sum_{j=1}^{m}(C_j\cos j\varphi + D_j\sin j\varphi) \end{cases}$$

$$\omega = P_0 + \sum_{j=1}^{m}(P_j\cos j\varphi + Q_j\sin j\varphi)$$

$$T = \int_0^{2\pi}\frac{\mathrm{d}\varphi}{\omega} \quad (28)$$

3 极限环的稳定和分岔

极限环的稳定性特征指数

$$\gamma = \frac{\varepsilon}{T}\int_0^{2\pi}\frac{1}{\omega}\{\alpha_1 + \beta_2 + b(2\alpha_3 + \beta_4) + b^2(3\alpha_6 + \alpha_7) + a[2\alpha_3 + \beta_4 + 2b(3\alpha_6 + \alpha_7)]\cos\varphi + a^2(3\alpha_6 + \alpha_7)\cos^2\varphi + [\alpha_4 + 2\beta_5 + 2(b + a\cos\varphi)(\alpha_7 + \alpha_8)]y + (\alpha_8 + 3\alpha_9)y^2\}\mathrm{d}\varphi \quad (29)$$

将式（29）中$\frac{1}{\omega}$及大括号内的式子（记为F）分别富氏展开为

$$\frac{1}{\omega} = \bar{P}_0 + \sum_{k=1}^{m}(\bar{P}_k\cos k\varphi + \bar{Q}_k\sin k\varphi) \quad (30)$$

$$F = \bar{C}_0 + \sum_{j=1}^{m}(\bar{C}_j\cos j\varphi + \bar{D}_j\cos j\varphi) \quad (31)$$

则极限环的稳定性特征指数

$$\gamma = \frac{2\pi\varepsilon}{T}[\bar{C}_0\bar{P}_0 + 0.5\sum_{i=1}^{m}(\bar{C}_i\bar{P}_i + \bar{D}_i\bar{Q}_i)]$$

当$\gamma < 0(> 0)$时极限环稳定（不稳定）。

极限环的数目及其随参数的变化而变化的演变过程，可由式（9）和（10）定量给出。比如系统

（1）关于参数 β_2 与极限环振幅的关系曲线为

$$\beta_2 = \int_0^{2\pi}(y' - \beta_1 x - \beta_3 x^2 - \beta_4 xy - \beta_5 y^2 - \beta_6 x^3 - \beta_7 x^2 y - \beta_8 xy^2 - \beta_9 y^3)\,\mathrm{d}\varphi \Big/ \int_0^{2\pi} y\,\mathrm{d}\varphi \triangleq \beta_2(a) \quad (32)$$

通过曲线（32），可以直观了解极限环随参数变化而产生、稳定、分岔与消失的全过程。

将式（30）代入（28），系统的周期为

$$T = \int_0^{2\pi}\frac{\mathrm{d}\varphi}{\omega} = 2\pi\bar{P}_0$$

4 算例

计算平面三次多项式微分系统

$$\dot{x} = -0.3x + xy$$
$$\dot{y} = 2.5y^2(1 - x - y) - xy \quad (33)$$

的极限环。

系统（33）有两奇点（0，0）及（0.3，0.3）。绕前者附近无极限环，绕后者则有一稳定极限环。为了用本文给出的方法计算之，引进变换

$$x = \bar{x} + 0.3, y = \bar{y} + 0.3$$

代入式（33），为方便，仍以 x, y 记 $\bar{x}, \bar{y}$，有

$$\dot{x} = 0.3y + \varepsilon xy$$
$$\dot{y} = -0.525x + \varepsilon(0.075y - 2.5xy - 0.5y^2 - 2.5xy^2 - 2.5y^3) \quad (34)$$

式（34）中，$\varepsilon = 1$。

当 $\varepsilon \approx 0$ 时，由式（12）至（16）得极限环的零阶近似解为

$$a_0 = 0.3024, b_0 = 0, \omega_0 = 0.3969,$$
$$x_0 = a_0\cos\varphi, y_0 = -\frac{a_0}{\alpha_2}\omega_0\sin\varphi$$

取 $\Delta\varepsilon = 0.1, m = 6$，经 10 次参数增量步骤，系统（34）近似极限环的振幅、偏心距、周期及稳定指数分别为

$$a = 0.193341, b = -0.025847,$$
$$T = 19.821889, \gamma = -0.011002$$

相关富氏系数见表 1。稳定极限环相图见图 1。系统（33）近似极限环和频率的表达式及周期为

$$x = 0.3 + 0.19334\cos\varphi - 0.02585$$
$$y = 0.3 + \sum_{j=0}^{6}(C_j\cos j\varphi + D_j\sin j\varphi)$$
$$\omega = \sum_{j=0}^{6}(P_j\cos j\varphi + Q_j\sin j\varphi)$$
$$T = 19.821889$$

表 1 极限环的富氏系数

Table 1 Fourier coefficient of the limit cycle

C	D	P	Q
0.061 201	0.0	0.357 633	0.0
-0.036 500	-0.293 719	0.071 608	-0.137 063
-0.041 965	0.081 614	-0.036 089	0.023 075
0.026 583	-0.023 383	-0.014 044	-0.008 133
-0.014 397	0.010 070	-0.010 718	0.004 050
0.007 638	-0.000 309	-0.009 176	-0.001 525
-0.003 231	0.003 345	0.001 819	0.000 018

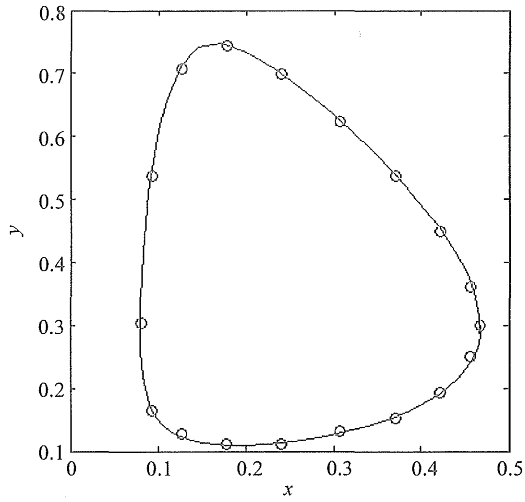

图 1 系统（33）的极限环

Fig. 1 Limit cycle of system (33)

°本文方法；—数值积分法

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无人机各机载设备均能正常工作。在该型无人机飞行过程中，各地面雷达和发射设备使用正常功率对其直接照射（距离 5 km），无人机仍能正常接收指令并根据要求完成各项任务，证明经过系统级电磁兼容测试的无人机能够在实际战场电磁环境下正常工作。

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